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BEHAVIORAL ANALYSIS OF LWR MODEL UNDER DIFFERENT EQUILIIBRIUM VELOCITY DISTRIBUTIONS

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ABSTRACT: The Lighthil-Whitham-Richards (LWR) model characterizes traffic flow based on the conservation of vehicles. Changes in velocities are based on the changes in densities which is given by equilibrium velocity distributions. The LWR model is employed with target to attain the equilibrium velocity. In this paper, density behavior of the LWR model is investigated using different equilibrium velocity distributions. The numerical solution of LWR model is obtained by applying the first order upwind scheme (FOUS). The stability of the numerical scheme is achieved by the Courant-Friedrich-Lewy (CFL) conditions. The performance results show that the density behavior of LWR model is realistic and smooth with target to attain the Greenshields, Underwood and Northwestern equilibrium velocity. Whereas, with target to attain the Greenberg velocity, the density behavior of LWR model is not realistic, and the variations in density grow over time.

Keywords: Lighthill-Whitham-Richards (LWR) model, Equilibrium velocity distributions, Courant-Friedrich-Lewy (CFL) conditions, First Order Upwind Scheme (FOUS).

INTRODUCTION

Rise in population coupled with industrial and commercial growth resulted in rapid increase in traffic demand (Button *et al.*, 2004). When traffic demand exceeds the road capacity, it causes traffic congestion (Imran, 2018), which results in traffic delays, lessen traffic safety and increases vehicle emissions. To limit the increase in travel time and keep cities livable, traffic congestion could essentially be reduced by adapting control strategies such as encouraging people to use different modes of transport, travel at different time or have a choice of choosing different traffic routes. The traffic control strategies can be employed, if the nature and behavior of traffic flow is predetermined. That is, where and how the traffic congestion is caused. The aforesaid assessment is supported by traffic flow models which predict and describe the traffic flow (Kessels, 2018).

Traffic flow models are classified as macroscopic when the traffic velocity, density and flow is treated cumulatively (Khan and Gulliver, 2019). These models are less computationally complex (Khan and Gulliver, 2018). In macroscopic domain the traffic flow is similar to the fluid flow. The behavior of fluid flow is described by the integral or differential systems (Maciejewski, 2018). The solution of these models give flow rates, densities and velocities over the road. The traffic behavior over the road can be determined with time evolution (Khan and Gulliver, 2019).

Greenshields in 1934 carried out study of the traffic flow, and laid the foundation of traffic flow studies. Later, in 1935 Greenshields proposed a linear relation between velocity and density (Greenshields *et al.*, 1935). Following the Greenshields developments, Greenberg considered a logarithmic $v - \rho$ relationship (Greenberg, 1959), however, the free flow velocity of the Greenberg model tends to infinity at low densities which is not realistic. Underwood employed an exponential relationship between the velocity and density (Underwood, 1961). The limitation of this model is that it predicts zero velocity for large densities. The Greenshields model has been modified to provide a more realistic characterization of traffic flow (Munjal and Pipes, 1971; Wang *et al.*, 2011; Lighthill and Whitham, 1955; Richards, 1956), proposed a macroscopic traffic flow model, the LWR model (Lighthill and Whitham, 1955; Richards, 1956) which is based on vehicle's conservation on a road which can be characterized by temporal changes in traffic density and spatial changes in flow. The LWR model is the simple to implement (Whitham, 1971) which can characterize small changes in traffic flow (Technical, 2015; Coscia, 2004). The LWR model is given as

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v(\rho)}{\partial x} = 0. \quad (1)$$

Where ρ is the traffic density and $v(\rho)$ is the equilibrium velocity. Traffic is adjusted according to the equilibrium velocity distribution. The relation between velocity and density has impact on ameliorating and predicting the traffic flow.

In this paper the LWR model density behavior is analyzed under different equilibrium velocity distribution models. When changes in density occur changes in velocity occur, the trend of change in velocity according to change in density is followed linearly in Greenshields model. When the LWR model is employed with target to attain Greenshields equilibrium velocity, the density behavior with the LWR model is different to that of the other models which define $v-\rho$ relation exponentially or logarithmically. The $v-\rho$ relation is usually defined depending upon the traffic conditions. In heterogeneous traffic a linear $v-\rho$ relation may not predict accurate density behavior as the change in velocity according to density does not follow a linear relation. Comprehensive analysis of the density behavior with LWR model for four different v-ρ relations is presented. The LWR model is numerically discretized by first order upwind scheme (FOUS) and the stability of the numerical scheme is ensured by Courant, Friedrich and Lewy (CFL) conditions [22].

The remainder of this paper is organized as follows. The traffic models are presented in Section 2. The numerical solution and stability analysis are given in Section 3. The performance of the LWR model is given in Section 4 and finally conclusions are given in Section 5.

MATERIALS AND METHODS

TRAFFIC FLOW MODELS: *Equilibrium Velocity Distribution Models:* Flow q, velocity v and density ρ are the macroscopic traffic variable (Ni, 2016). In the presence of small distance headways, density is large and high velocity seldom occurs. Similarly, high velocity occurs with low densities. Greenshields in 1934 studied the relation between velocity and distance between the vehicles. The relation was graphically presented, which is called the fundamental diagram of traffic (Kessels, 2018).

Greenshields in 1935 from his studies proposed a linear relation between velocity and density, which is given as

$$v(\rho)=v_m\left(1-\frac{\rho}{\rho_m}\right). \qquad (2)$$

Where v_m is the maximum velocity, ρ is average density and ρ_m is the maximum or jam density. $v(\rho)$ represents velocity as a function of density. Greenshields assumption are not entirely perfect but relatively simple and fairly accurate. After the Greenshields developments many models were presented for $v-\rho$ relationhip, which are given in Table 1.

Table 1: Equilibrium velocity distributions.

Single Regime Model	Function	Parameters
Greenshields Model 1935	$v(\rho)=v_m(1-\frac{\rho}{\rho_m})$	v_m, ρ_m
Greenberg Model 1959	$v(\rho)=v_m\log(\frac{\rho_m}{\rho})$	v_m, ρ_m
Underwood Model 1961	$v(\rho)=v_m\exp(\frac{\rho}{\rho_m})$	v_m, ρ_m
Northwestern Model 1967	$v(\rho)=v_m exp[-\frac{1}{2}(\frac{\rho}{\rho_m})]^2$	v_m, ρ_m,

The Lighthill, Whitham and Richards (LWR) Model: The macroscopic Lighthill-Whitham-Richards (LWR) traffic flow model was presented independently by Lighthill, Whitham and Richards in 1955-56, which is based on the conservation of vehicles. Consider a road section with two traffic counting stations, A and B with no traffic sources and sinks. That is no traffic enters or leaves between the counting stations. N_i and q_i are the number of vehicles and flow passing station i, respectively, in time Δt. Δx is the distance between the stations. When the number of vehicles entering station A is greater than the number of vehicles leaving station B,

$$N_A>N_B, \quad (3)$$

this implies that there is a buildup of vehicles between the stations as there is no sink between the stations, The change in number of vehicles between the stations is ΔN which is given as

$$\Delta N=N_B-N_A, \quad (4)$$

the buildup ΔN between the stations will be negative. Number of vehicles passing a point in a specific time is flow q, which is given as

$$q=\frac{\rho}{t}. \quad (5)$$

Flow at station A is q_A and at station B is q_B, then

$$q_A=\frac{N_A}{\Delta t},(6)$$

and

$$q_B=\frac{N_B}{\Delta t}.(7)$$

When the flow q_A is different from flow q_B, the change in flow Δq is

$$\Delta q=\frac{\Delta N}{\Delta t}. \qquad (8)$$

From (8), ΔN between the stations is

$$\Delta N=\Delta q\Delta t, \qquad (9)$$

and the change in number of vehicles is negative, then

$$\Delta N=-\Delta q\Delta t. \quad (10)$$

Density ρ is the number of vehicles per unit length. $\Delta\rho$ is the change in number of vehicles between the station A and B, thus

$$\Delta\rho=-\frac{N_B-N_A}{\Delta x}. \quad (11)$$

From (11) the changes in number of vehicles is

$$-\Delta N=\Delta\rho\Delta x. \quad (12)$$

Under the assumption of conservation of vehicles (Kacharoo, 2009), (10) and (12) gives,

$$-\Delta q\Delta t=\Delta\rho\Delta x. \ (13)$$

Rearranging (13), gives

$$-\frac{\Delta q}{\Delta x}=\frac{\Delta\rho}{\Delta t}, \qquad (14)$$

as the system is conserved, thus

$$\frac{\Delta q}{\Delta x} + \frac{\Delta \rho}{\Delta t} = 0. \qquad (15)$$

Since the finite increments are allowed to come from infinitesimal in the limit, then (15) is expressed as

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0. \qquad (16)$$

as

$$q = \rho v, \ (17)$$

substituting (17), (16) gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0, \qquad (18)$$

where v is velocity and ρ is density. Since, Velocity is the function of density, thus

$$v = v(\rho). \qquad (19)$$

Substituing (19), (18) takes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v(\rho)}{\partial x} = 0, \ (20)$$

which is the LWR model.

Numerical solution of the macroscopic traffic flow models are obtained by numerical approximations as they are systems of partial differential equation (Trieber and Kesting, 2013). The numerical solution of the LWR model is obtained using the first order upwind scheme (FOUS) (LeVeque, 2007). The solution space is splitted spatially and temporally to get a uniform computational grid. Δx is the width of a road segment which is the difference between two consecutive points in the x direction while Δt is the time step in t direction. The density is approximations are made over equidistant road segments $(x_i + \Delta x, x_i - \Delta x)$. Then, it is approximated over the time interval (t_{n+1}, t_n), where $t_{n+1} - t_n = \Delta t$.

The LWR model is approximated by spatial derivatives of flow and temporal derivatives of density, respectively. The forward in time density approximation is

$$\frac{\partial \rho(t_n, x_i)}{\partial t} = \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t}, \qquad (21)$$

and backward in space flow approximation is

$$\frac{\partial q(t_n, x_i)}{\partial x} = \frac{q_i^n - q_{i-1}^n}{\Delta x}. \qquad (22)$$

Substituting (21) and (22) in (1) gives

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\left(q(\rho_i^n) - q(\rho_{i-1}^n)\right), \qquad (23)$$

where the flux $q(\rho_i^n)$ for the LWR model at the i-th time step and n-th road segment is

$$\rho_i^n v(\rho_i^n), \qquad (24)$$

and the flux $q(\rho_{i-1}^n)$ at the $(i-1)$-th time step and n-th road segment is

$$\rho_{i-1}^n v(\rho_{i-1}^n). \qquad (25)$$

Then the density at the i-th time step and $(n+1)$-th road segment with the LWR model is approximated as

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\left(\rho_i^n v(\rho_i^n) - \rho_{i-1}^n v(\rho_{i-1}^n)\right). \ (26)$$

A numerical technique should be stable (Causon, 2010). The stability of the scheme can be guaranteed by choosing a suitable time step Δt. The traffic covers a maximum distance of $v_m \Delta t$ in a single time step. To approximate the density in this time, the distance covered by the traffic should be less than a road segment so that

$$v_m \Delta t \leq \Delta x, \qquad (27)$$

and rearranging (27), gives

$$\Delta t \leq \frac{\Delta x}{v_m}. (28)$$

This provides the maximum allowable time step Δt. In this case, the slope of the gradient is always less than or equal to 1, which is known as the Courant number given by

$$c = v_m \frac{\Delta t}{\Delta x}. \qquad (29)$$

$c \leq 1$ is known as the Courant, Friedrich and Lewy (CFL) condition for the numerical stability of the scheme (de Moura, 2013). The maximum velocity for the LWR model is $v_m = 16.66$ m/s, and the time and roads steps are $\Delta t = 0.09$ s and $\Delta x = 2$ m, respectively. The CFL condition gives

$$c = 16.66 \times \frac{0.09}{2} = 0.74 \leq 1, (30)$$

so the numerical solutions are stable.

RESULTS AND DISCUSSION

The performance of the LWR model is evaluated over a 400 m straight road section. The simulation parameters are given in Table 2, the total simulation time is 40 s, and the time and road steps are 0.09 s and 2 m, respectively. The maximum normalized density is 1 and the maximum velocity is 16.66 m/s and the target equilibrium velocity distribution is Greensields, Greenberg, Underwood and Northwestern as given in Table 1.

Table 2: Simulation Parameters.

Description	Value
Simulation time	40 s
Road length	400 m
Maximum velocity, $\boldsymbol{v_m}$	16.66 m/s
Time step	0.09 s
Road step	2 m
Total time steps	445
Total road steps	200
Equilibrium velocity distribution	See Table 1
Maximum density, $\boldsymbol{\rho_m}$	1

The initial density distribution is given in Figure 1. The density is 0.1 at 1 m, increases to 0.193 at 44 m, decreases to 0.006 at 130 m, and then increases to 0.193 at 218 m. Density is 0.006 and 0.193 at 308 and 400, respectively.

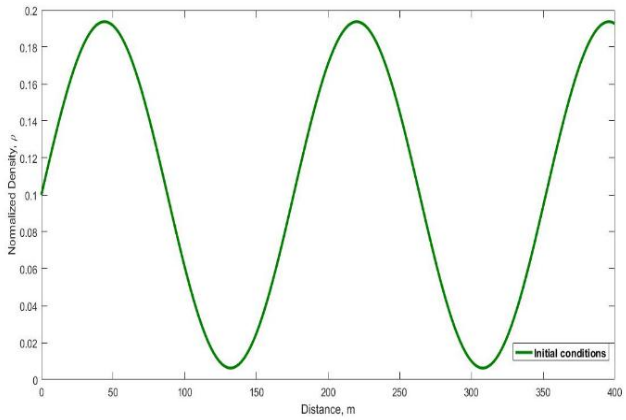

Figure 1: Initial density distribution over a 400 m road section.

The density behavior of the LWR model at 1 s, 20 s and 40 s over a 400 m road section with target to achieve the Greenshields equilibrium velocity is given in Figure 2. At 1 s, the traffic density is approximately 0 at 1 m, increases to 0.193 at 44 m. At 146 m, density decreases to 0.05 which then increases to 0.193 at 220 m. Density is 0.05 and 0.192 at 322 m and 400 m, respectively. At 20 s, between 0 m and 170 m, density 0 is uniform as the traffic moves forward and no ingress of vehicles is allowed due to the non periodic boundary conditions. Density is 0.124 at 180 m, decreases to 0.058 at 310 m. At 320 m and 400 m, density is 0.142 and 0.10, respectively. At 40 s, density is 0.09 at 394 m and 400 m. Density propagation smooth.

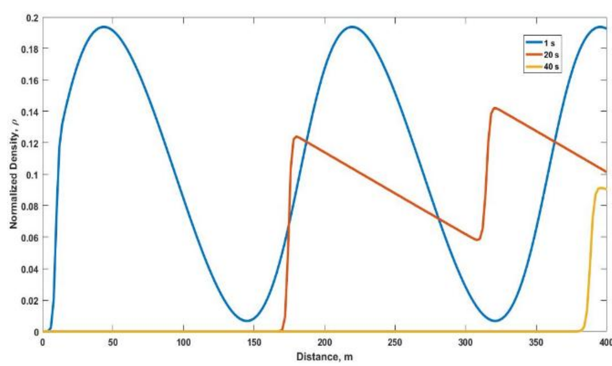

Figure 2: Traffic density behavior with LWR model with target to achieve the Greenshields equilibrium velocity over a 400 m road section at 1 s, 20 s and 40 s.

The density behavior of the LWR model at 1 s, 20 s and 35 s over a 400 m road section with target to achieve the Underwood equilibrium velocity is given in Figure 3. At 1 s, density increases to 0.193 at 48 m from 0 at 1 m. At 146 m, density is 0.05, increases to 0.193 at 224 m. The density then decreases to 0.05 at 322 m and increases to 0.193 at 400 m. At 20 s, density is 0 between 0 m and 188 m. At 200 m, density increases to 0.142 which then decreases to 0.05 at 340 m. Density is 0.16 and 0.125 at 354 m and 400 m, respectively. At 35 s, density reaches 0.09 at 400 m. Results show that density propagation of LWR model with target to achieve the Underwood equilibrium velocity is faster than the LWR model with Greenshields model.

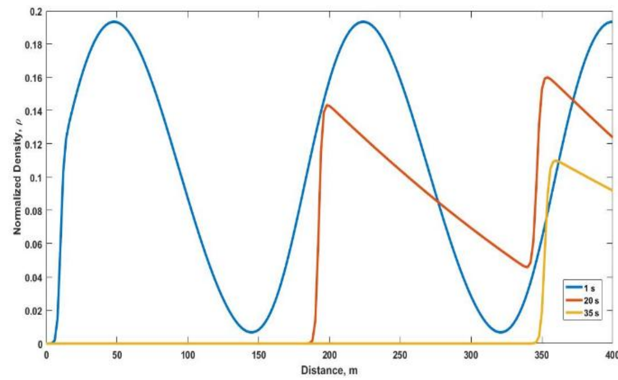

Figure 3: Traffic density behavior with LWR model with target to achieve the Underwood equilibrium velocity over a 400 m road section at 1 s, 20 s and 40 s.

The density behavior of the LWR model at 1 s, 20 s and 29 s over a 400 m road section with target to achieve the Northwestern equilibrium velocity is given in Figure 4. At 1 s, density increases to 0.192 at 52 m from 0 at 8 m. At 148 m, density decreases to 0.06, and increases to 0.193 at 228 m. Density is 0.05 and 0.192 at 322 m and 400 m. At 20 s, density is 0.17 and 0.04 at 278 m and 400 m, respectively. At 29 s, density reaches 0.148 at 400 m. Density propagation with LWR model is quicker with Northwestern model than LWR model with Underwood and Greenshields equilibrium velocity distributions. Density behavior is normal within plausible limits.

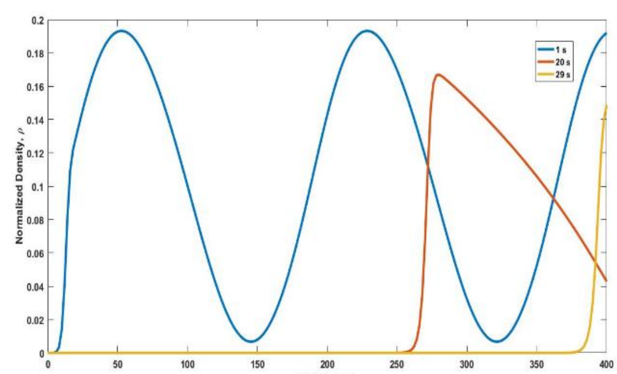

Figure 4: Traffic density behavior with LWR model with target to achieve the Northwestern equilibrium velocity over a 400 m road section at 1 s, 20 s and 40 s.

The density behavior of the LWR model at 0.09 s and 1 s over a 400 m road section with target to achieve the Greenberg equilibrium velocity is given in Figure 5. Density propagation is same as the initial density distribution in Fig 1. At 1 s, density is 0.16 at 18 m with slight increase in density to 0.19 at 38 m. At 152 m, density is 0.01, between 152 m and 174 m, the density behavior is oscillatory, reaches below 0. The density behavior between 328 m and 350 m is again oscillatory and not realistic, at 400 m, density is 0.19.

It is evident from the results that the density propagation of LWR model with Northwestern model is faster than Underwood model and Greenshields model. The density propagation is faster with Underwood model than Greenshields model. Conversely, the LWR model density propagation with Greenberg model is not realistic as it produces oscillatory behavior which grow over time, this is because the free flow velocity of the Greenberg model tends to infinity when low densities occur.

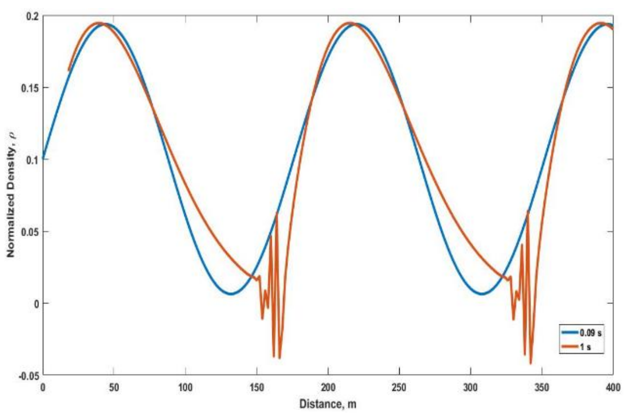

Figure 5: Traffic density behavior with LWR model with target to achieve the Greenberg equilibrium velocity over a 400 m road section at 1 s, 20 s and 40 s.

Conclusion: The LWR model is a macroscopic traffic flow model which is based on the conservation of vehicles. This model is employed with target to attain equilibrium velocity which is given by $v - \rho$ distribution models. In this paper, the density behavior of LWR model is analyzed under different equilibrium velocity distribution models to determine the favorable models. With Greenshields, Underwood and Northwestern models the LWR model density behavior is realistic and smooth. Conversely, the density behavior of LWR model to attain Greenberg equilibrium velocity is not realistic. The variation in traffic densities grow over time, and is not a realistic traffic behavior.

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