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TOTAL VARIATION AND FRACTIONAL ORDER BASED MODEL FOR IMAGE RESTORATION

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ABSTRACT: This paper introduces a new fractional order based total variation model for the removal of multiplicative noise. The resultant Euler Lagrange partial differential equation (PDE) associated with minimization functional of FTV energy model is usually obtained for image restoration. The variational model minimization problem is achieved by the exploitation time marching scheme. These results of the proposed work clearly indicate that the model for multiplicative noise removal not only significantly removes the noise of multiplicative type but also it reduces the staircase effect more effectively than all other models previously reported for noise removal. The proposed model is good in image denoising and increases the peak signal to noise ratio compared with integer order based model and TV-based models.

Keywords: Fractional Order Derivative; Noise Removal; Total Variation (TV); Multiplicative Noise.

INTRODUCTION

Image denoising is a research topic and is mostly used in image process computer vision fields. There are many types of noises but the most commonly used noises are multiplicative and additive. There are numbers of method which have been designed for the removal of all such types of noise problems. Some of these approaches are mentioned here which are widely used all over the world for noise removal both from signals and images, for instance, wavelet schemes, stochastic schemes, and variational schemes.

Multiplicative is one of most important noise removal task and we will focus on the removal of multiplicative noise removal problem. This study also includes the comparative studies which are describing new and old models for multiplicative noise removal and to check the results and show the performance of new multiplicative noise removal model.

Non- local field theories are doing lot much better nowadays just because of the growing field of mathematics which is known as fractional calculus. This field of mathematics has greatly invaded the field of computer vision. The main reason behind the successful development of the fractional calculus is that researchers were expecting that fractional calculus will lead to a more elegant and effective way of treating problems of blocky effect and fine scale features. In the last few years, the idea of fractional operator and measure has been deeply studied in many engineering and sciences models. Fractional order derivative operators have found good tools in image processing to deal with fine scale information such as edge and texture. In literature, the use of fractional order total variation (FOTV) based models have proven an important tool and has an essential role in many image processing applications such as image de-noising, in-painting and motion estimation (Zhang *et al.*, 2014). Furthermore, the long term memory is an important feature of fractional order differentiation and is a dominant difference between fractional and integer order differentiation that can be seen in (Hu and Hu, 2015; Chen *et al.*, 2013).

In this study, we have presented a new model for multiplicative noise removal, in which integer order total variation based model is modifying the regularization term by replacing the gradient of u with fractional order derivative and hence obtained new fidelity term which results in the novel model. The application of this new model is to obtain good restoration results and reduce the blocky effect (Bagchi *et al.*, 2016, Chen *et al.*, 2013).

MATERIALS AND METHODS

Consider the model used for multiplicative noise $f = u + \sqrt{u\eta}$, with given observed image f, true image u, and η represents the multiplicative noise under Rayleigh distribution. Inspired by the work done in (Hu and Hu, 2015), the logarithmic amplification results in $g = \log u$ and hence the following minimization functional is obtained.

$$\min_u \left\{ E(u) = R(u) + \lambda \int_\Omega \frac{f}{2} (f^2 e^{-g} + g - 2f + e^g) + g) \, dxdy \right\} \qquad (1)$$

$$R(u)=\int_{\Omega}\left|\nabla^{\alpha}(u)\right|dxdy$$

Where and $\left|\nabla^{\alpha}(u)\right|=\sqrt{(\nabla_{x}^{\alpha}u)^{2}+(\nabla_{y}^{\alpha}u)^{2}+\varepsilon^{2}}$. In (1), the first term shows the regularization term while the second terms represent the data fidelity term.

The associated Euler-Lagrange equation from functional (1) is written as under.

$$(-1)^{\alpha}\nabla^{\alpha}.\left(\frac{\nabla^{\alpha}u}{\left|\nabla^{\alpha}u\right|^{2}+\varepsilon^{2}}\right)+\lambda\left(\frac{f}{2}(1+e^{g}-f^{2}e^{-g})+1=0, in\,\Omega,\right\}$$

$$\frac{\partial u}{\partial\vec{n}}=0\ on\ \partial\Omega \qquad (2)$$

As $u\succ 0,$ so the Euler-Lagrange equation (2) can be rewritten as under.

$$(-1)^{\alpha}\left[\nabla^{\alpha}.\left(\frac{\nabla^{\alpha}u}{\sqrt{\left|\nabla^{\alpha}u\right|^{2}+\varepsilon^{2}}}\right)\right]+\lambda\left(\frac{f}{2}(1+e^{g}-f^{2}e^{-g})+1\right)=0 in\Omega,\}$$

$$\frac{\partial u}{\partial\vec{n}}=0\ on\ \partial\Omega \qquad (3)$$

Equation (3) can also be redefined as follow.

$$(-1)^{\alpha}[\overline{\nabla_{x}^{\alpha}}.\left(\frac{\nabla_{x}^{\alpha}u}{\sqrt{\left|\nabla^{\alpha}u\right|^{2}+\varepsilon^{2}}}\right)+\overline{\nabla_{y}^{\alpha}}.\left(\frac{\nabla_{y}^{\alpha}u}{\sqrt{\left|\nabla^{\alpha}u\right|^{2}+\varepsilon^{2}}}\right)$$

$$+\lambda\left(\frac{f}{2}(1+e^{g}-f^{2}e^{-g})+1\right)=0 in\,\Omega,\}$$

$$\frac{\partial u}{\partial\vec{n}}=0\ on\partial\ \Omega \qquad (4)$$

Then the time dependent parabolic equation is applied to solve nonlinear PDE equation (4) which is written as under.

$$u_{t}=(-1)^{\alpha}[\overline{\nabla_{x}^{\alpha}}.\left(\frac{\nabla_{x}^{\alpha}u}{\sqrt{\left|\nabla^{\alpha}u\right|^{2}+\varepsilon^{2}}}\right)+$$

$$\overline{\nabla_{y}^{\alpha}}.\left(\frac{\nabla_{y}^{\alpha}u}{\sqrt{\left|\nabla^{\alpha}u\right|^{2}+\varepsilon^{2}}}\right)+$$

$$\lambda\left(\frac{f}{2}(1+e^{g}-f^{2}e^{-g})+1\right)\}$$

$$for\ t\succ 0,(x,y)\in\Omega. \qquad (5)$$

In this model the regularization parameter $\varepsilon\succ 0$ is selected as approaches to 0. The proposed model (1) will become integer order model if $\alpha=1$ is selected.

Numerical Methods: The iterative scheme maybe utilized to solve the PDE equation (5) numerically. Suppose the time step size ∂t and space gird of size h, so the approximation of time and space is defined as under.

$$t=n\partial t, x=ih, y=jh, n=0,1,2,........., M-1. \qquad (6)$$

For numerical experiments, we chose $h=1$ and $\varepsilon=10^{-4}$. The differential equation (5) is approximated numerically by the following equation.

$$u_{i,j}^{n+1}=u_{i,j}^{n}+\partial t[(-1)^{\alpha}[\overline{\nabla_{x}^{\alpha}}.\left(\frac{\nabla_{x}^{\alpha}u_{i,j}}{\sqrt{\left|\nabla^{\alpha}u_{i,j}\right|^{2}+\varepsilon^{2}}}\right)$$

$$+\overline{\nabla_{y}^{\alpha}}.\left(\frac{\nabla_{y}^{\alpha}u_{i,j}}{\sqrt{\left|\nabla^{\alpha}u_{i,j}\right|^{2}+\varepsilon^{2}}}\right)+\lambda\left(\frac{f_{i,j}}{2}(1+e^{g_{i,g}}\right.$$

$$\left.-f_{i,j}{}^{2}e^{-g_{i,j}})+1\right)\} \qquad (7)$$

Assume that the digital image $u(i,j)\big|_{i,j=1}^{N}$, where $u(i,j)=0$ for $i,j\succ N$ or $i,j\prec 1$. The discrete fractional difference based numerical scheme is utilized to figure out the fractional order derivative rapidly and stably, which is given as under.

$$\overline{\nabla}_{x}^{\alpha}\phi_{i,j}=\sum_{k=0}^{K-1}[C_{1}^{\alpha}(k)u(i-k,j)], \overline{\nabla}_{y}^{\alpha}u_{i,j}=\sum_{k=0}^{K-1}[C_{1}^{\alpha}(k)u(i,j-k)] \qquad (8)$$

$$\nabla_{x}^{\alpha}\phi_{i,j}=\sum_{k=0}^{K-1}[C_{1}^{\alpha}(k)u(i+k,j)], u_{i,j}=\nabla_{y}^{\alpha}\sum_{k=0}^{K-1}[C_{1}^{\alpha}(k)u(i,j+k)] \qquad (9)$$

where $C_{1}^{\alpha}(k)=(-1)^{\alpha}\frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}$ shows the polynomial coefficients $(1-x)^{\alpha}$. The coefficients can also be found out by the following ways.

$$C_1^{\alpha}(0)=1, C_1^{\alpha}(k)=(1-\frac{\alpha+1}{k})C_1^{\alpha}(k-1), k=1,2,3........K.$$

The selection of the step size ∂t is a complicated issue that is responsible for quick convergence due to the nonlinearity of the equation and hence the selection of the optimal value of step size is hard to choose theoretically and computationally expensive. Alternatively, we have observed that the best value of step size throughout our experimental tests is 10^{-3} . The proposed numerical scheme is explained by the following algorithm.

Algorithm-1 Explicit Time Marching Algorithm (ETMA)

1. Procedure: Let the input image be f such that $u^{(0)}(i,j)=f(i,j), \alpha\in[1,2], \lambda\succ 0, \partial t=10^{-3}, \varepsilon=0.0001.$

2. Let $u_0^n=u^{n-1}$. Numerically compute $u^{(n+1)}$ as following,

3. $$u_{i,j}^{n+1}=u_{i,j}^{(n)}+\partial t[(-1)^{\alpha}[\overline{\nabla_x^{\alpha}}.(\frac{\nabla_x^{\alpha}u_{i,j}^{(n)}}{\sqrt{\left|\nabla^{\alpha}u^{(n)}{}_{i,j}\right|^2+\varepsilon^2}})+\overline{\nabla_y^{\alpha}}.(\frac{\nabla_y^{\alpha}u^{(n)}{}_{i,j}}{\sqrt{\left|\nabla^{\alpha}u^{(n)}{}_{i,j}\right|^2+\varepsilon^2}})+\lambda(\frac{f_{i,j}}{2}(1+e^{g^{(n)}{}_{i,g}}-f^2{}_{i,j}{}^2e^{-g^{(n)}{}_{i,j}})+1],$$

4. $with u=e^g.$

Checking stopping criteria that is $\frac{\left\|\hat{u}-u\right\|^2}{\|u\|^2}\leq Tol;$ else ways $n=n+1$. Result output $u=u^{(n+1)}$.
End Procedure.

RESULTS AND DISCUSSION

The peak signal to noise ratio (PSNR) is used as a tool to check and measure the image restoration performance of all used schemes.

$$PSNR=10\log_{10}[\frac{(m_1xm_2)\max(u)^2}{\|f-u\|^2}], \quad (11)$$

In the above given equation f describes the true picture, u denotes the image being restored and m_1xm_2 specifies the size of the given image. The restoration performance is directly related to that of the High PSNR values, higher the PSNR value much better will be the restoration performance. House image, Lena image and Medical images have been corrupted by multiplicative noise and these images are known as gray scale images. As all the images are accumulated up of pixels and every of those pixels are interrupted or degraded via a noise which follows a Gamma distribution having different variances (noise variance $\sigma^2=0.02, 0.03$ *and* 0.05) which are displayed in the given figure.

The proper selection of the fractional order parameter $\alpha\in[1,2]$ and the range of values for $\lambda\in[10,35]$ depend on the image to de-noise. Some parameters in this research experiment were taken as optimal parameters in order to ensure to have better results in denoising process.

Consider the restoration of Lena image contaminated by multiplicative noise with $\sigma^2=0.02$ respectively. De-noised images are shown in figure No 1. The experimental process demonstrates that image restoration (PSNR) performance of proposed fractional order total variation (FOTV) based model is superior to integer order total variation (IOTV) based model

In this case we have considered a real image namely House of size $(256)^2$ pixels with $\sigma^2=0.03$. Results are displayed in figure 2. It shows the restoration images by the new model. The results shown that the de-noising performance of the fractional order based new model is superior to integer order based 1.0 $M^{\alpha=1.0}$ model.

Fig. 1. Lena image of size $(256)^2$ Restoration results using the FOTV model M^{α} ,(a) , (f) Lena noiseless

image, (b) , (f) degraded image, (c) , (g) Restored image by the proposed model for $\alpha = 1$ *and* $\alpha = 1.5,$ (d), (h) Graph between number of iterations and PSNR.

Fig.2. House image $(256)^2$ Reconstructed results using new model M^{α} , (a) , (f) House true image, (b) , (f) Degraded image, (c) , (g) Reconstructed image by the new model for $\alpha = 1$ and $\alpha = 1.3$, (d), (h) Graph between number of iterations and PSNR.

We have taken test problem and applied real medical image of size $(256)^2$ pixels corrupted by multiplicative noise with $\sigma^2 = 0.05$. Results of algorithm-1 are shown in figure 3. Output results shown that the de-noising achievement of the fractional order model is superior to integer order based model.

Tab.1. Comparison of PSNR values, CPU times, and iterative numbers.

Image	**Size**				$M^{\alpha=1.0}$	M^{α}
		It.	**CPU**	α	**PSNR**	**PSNR**
Lena Image	256^2	20	4.62	α =1/1.5	27.21	28.82
House Image	256^2	20	3.96	α =1/1.3	24.90	25.46
Medical Image	256^2	20	4.69	α =1/1.4	28.23	29.72

The PSNR results shown in table 1 indicate the image restoration quality of the algorithm -1 used in the new (FOTV) based model is far better compared integer order total variation (IOTV) based scheme.

Total Variation Based Denoising Methods for Speckle Noises Images: The authors proposed a new algorithm (Bagchi *et al.*, 2016) for model (1) to remove the speckle noise from images. The minimization functional is given as under.

$$\min_u [\lambda J(u) + \int \frac{1}{2} \frac{|u-f|^2}{f} dx] \qquad (12)$$

The corresponding Euler Lagrange equation from functional (12) is given as

$$\partial J(u) + \frac{u-f}{\lambda u} \ni 0, \qquad (13)$$

From above equation (13) by MCSD algorithm, by setting $w = \dfrac{f-u}{\lambda u}$ and for w is supposed as the minimizer, the functional for w is given as under

$$\frac{\left\| \sqrt{uw} - \dfrac{f}{\lambda\sqrt{u}} \right\|^2}{2} + \frac{1}{2} J^*(w), \qquad (14)$$

Since $w \in K$, where $K = \{divp : p \in Y, \|p_{i,j}\| \le 1 \forall i, j\},$ and $J^* = H$, from the minimization of w to find p to recover u is determined as follow

$$u = f - \lambda u divp.$$

The MCSD algorithm is given as for $n \ge 0$ as given under of the gradient descent method as follows;

$$p_{i,j}^{n+1} = \frac{p_{i,j}^n + \tau(\nabla(\sqrt{u^n} divp^n - \dfrac{f}{(\lambda\sqrt{u^n})}))_{i,j}}{1 + \tau(\nabla(\sqrt{u^n} divp^n - \dfrac{f}{(\lambda\sqrt{u^n})}))_{i,j}},$$

$$\lambda^{n+1} = \frac{\| f - f_s \|}{\| u^n divp^{n+1} \|}, and$$

$$u^{n+1} = f - \lambda^{n+1} u^n divp^{n+1}$$

For further details, see (Bagchi *et al.*, 2016).

The displayed results in Figures 4 and Table 2 recommend that the proposed algorithm gives superior results compared to MCSD regarding the quality of restoration (SNR) for the same noise level and parameters as taken in model.

Fig.4. Lena image $(256)^2$ Obtained results using model M^{α} , (a) Lena true image, (b) Noisy image, (c), (g) Obtained image by the proposed model for $\alpha = 1.44$, (d) Graph between number of iterations and PSNR.

Tab. 2. It shows the PSNR results by algorithm-1 and MCSD on gray level images.

Image	Size		MCSD	M^{α}
		α	**PSNR**	**PSNR**
Lena Image	$(256)^2$	α =1.44	26.06	27.89

Table 2. It shows the PSNR results by algorithm-1 on gray level images. From the Table 2, we can see that the PSNR of the image restored by using the fractional order model is better than those restored by using the MCSD method.

Table 2 indicates that the image restoration (PSNR) performance of new fractional order model is better than that of MCSD model which the shows the good restoration performance of new model over MCSD model.

Conclusion: The main contributions of this research are illustrated as fallow one by one. Fractional order derivative definition was used for the removal of multiplicative noise using FOTV-image multiplicative noise removal model and was compared with that of IOTV based multiplicative noise removal model and their role in multiplicative noise removal model was also illustrated practically for denoising. Explicit time marching algorithm is employed for the associated non-linear Euler-Lagrange equation. The research results show that there was marked differences between result of FOTV model is compare to integer order based model. FOTV was better in noise removal especially in multiplicative cases than that of IOTV. In future work, we intend to apply multi-grid algorithm and other fast numerical methods to restore better degraded images.

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